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Homotopically Nontrivial Solutions for a Spherically Symmetric Gravitational Field

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The structure of kink solutions to the Einstein field equations is discussed and the most general form of spherically symmetric kink metric is constructed. A number of fluid solutions are investigated and an imperfect fluid solution with nonzero heat conduction is presented.

1. INTRODUCTION

The existence of kinks in general relativity may be understood in at least three ways: in terms of the homotopy classes of metrics $g_{\mu\nu}$—as described in the original work of Finkelstein and Misner (1959); in terms of deformation classes of direction fields on a Lorentz manifold $\mathcal{M}$—as described by Geroch and Horowitz (1979) and by Bugajska (1987); or in terms of inequivalent cross sections of the associated $GL(4, \mathbf{R})/SO(3, 1)$ bundle—as explained in the review article of Isham (1981). A central feature of a metric with kink(s) is that it cannot evolve or be transformed (by a *nonsingular* transformation) into a metric without a kink. One consequence of this is that Birkhoff's theorem (which states that a spherically symmetric gravitational field in empty space must be static) cannot be applied *globally* to a region where the number of kinks is nonzero. Hence, when considering the general expression for a spherically symmetric metric, the usual procedure of "transforming away" the g_{rt} cross term is inappropriate for a kink metric. Such a transformation would be singular, and so the metrics between which it transforms should be regarded as distinct. A similar situation arises in a recent study by Rosen (1983, 1985) of a spherically

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symmetric metric which satisfies the vacuum Einstein equations and yet is distinct from the Schwarzschild metric. Rosen's metric can be transformed into the Schwarzschild metric, but only by a transformation that possesses singularities. (However, although his metric has a nonzero g_{rt} term, it has no kink. The condition $g_{rt} \not\equiv 0$ is a necessary but not a sufficient condition for the existence of a kink.)

In the present paper, we shall consider the most general form of spherically symmetric metric. Since the decomposition of such a metric into its kink part and nonkink part has not appeared in the literature, this is done explicitly in Section 2. In a previous paper (Harriott and Williams, 1986), the authors introduced an energy-momentum tensor $T_{\mu\nu}$ for a fluid whose velocity 4-vector u^μ was defined in terms of a parameter that occurred naturally in a simple model of a kink metric. This definition is extended in Section 3 below, where a more general choice for u^μ is proposed. Section 4 presents some time-independent solutions of the Einstein equations. General forms for the Christoffel symbols, Ricci tensor, and various hydrodynamic quantities are listed in the Appendix.

The $(-+++)$ signature and the conventions of Misner *et al* (1973) are adopted throughout. Greek indices run over 0, 1, 2, 3 and Latin indices run over 1, 2, 3. The topology of the spacetime manifold is assumed to be trivial: $\mathcal{M} = \mathbf{R}^4$.

2. METRIC

The most general spherically symmetric metric is given in Cartesian coordinates $\{x^\mu\}$ by (Bergmann, 1942)

$$\begin{aligned} ds^2 &= g_{\mu\nu}\, dx^\mu\, dx^\nu \\ &= A\, dt^2 + 2B\chi_i\, dt\, dx^i + (C\delta_{ij} + D\chi_i\chi_j)\, dx^i\, dx^j \end{aligned}$$

where A, B, C, and D are functions of $t = x^0$ and $r = (\sum x^i x^i)^{1/2}$, and where $\chi_i = x^i/r$. This line element does not change its form under spatial rotation of the coordinates $\{x^i\}$.

A well-known result of matrix theory states that any nonsingular matrix can be written uniquely as the product of a symmetric, positive-definite matrix S and an orthogonal matrix Q. If this is done for a general relativistic metric $g_{\mu\nu}$,

$$\|g_{\mu\nu}\| = SQ$$

then, following Steenrod (1951), the signature condition on $g_{\mu\nu}$ implies that $SQ = QS$ and that Q is symmetric as well as orthogonal and that it forms a group isomorphic with $SO(3)$. [It is the "wrapping" of physical space, $\mathbf{R}^3$, about this $SO(3)$ that produces the kinks (Finkelstein, 1978; Shastri *et al.*, 1980)]. The set of 4×4 positive-definite matrices is *contractible* since the identity map $S \to S$ is homotopic to the constant map $S \to I$ through the homotopy

$$tI + (1-t)S, \qquad 0 \leq t \leq 1$$

Being positive definite, S is thus homotopically trivial. The SQ factorization of the metric can be thought of as the product of a kink part Q and a nonkink part S. The matrices S and Q will now be determined for the spherically symmetric metric given above.

According to Perlis (1952),

$$S = (\|g_{\mu\nu}\|^2)^{1/2}, \qquad Q = \|g_{\mu\nu}\| S^{-1}$$

To perform the square root operation, the metric must first be diagonalized. Since the metric is symmetric, there exists an orthogonal matrix P such that $P^T \|g_{\mu\nu}\| P = \mathrm{diag}(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$, with $\lambda_0 < 0$ and $\lambda_1, \lambda_2, \lambda_3 > 0$. Clearly,

$$S = P\,\mathrm{diag}(|\lambda_0|, \lambda_1, \lambda_2, \lambda_3) P^T, \qquad Q = P\,\mathrm{diag}(-1, 1, 1, 1) P^T$$

The eigenvalues are found to be

$$\lambda_0 = \{A + C + D - [(A - C - D)^2 + 4B^2]^{1/2}\}/2$$
$$\lambda_1 = \lambda_2 = C$$
$$\lambda_3 = \{A + C + D + [(A - C - D)^2 + 4B^2]^{1/2}\}/2$$

and, because λ_0 must be negative and $\lambda_1, \lambda_2, \lambda_3$ must be positive, the following inequalities must hold:

$$C > 0, \qquad B^2 > A(C + D)$$

The orthogonal matrix P can be deduced by the standard procedure, and is found to be

$$P = \begin{pmatrix} -(C+D-\lambda_0)/R_0 & 0 & 0 & -(C+D-\lambda_3)/R_3 \\ Bx^1/(rR_0) & 0 & -S_{23}/r & Bx^1/(rR_3) \\ Bx^2/(rR_0) & x^3/S_{23} & x^1x^2/(rS_{23}) & Bx^2/(rR_3) \\ Bx^3/(rR_0) & -x^2/S_{23} & x^1x^3/(rS_{23}) & Bx^3/(rR_3) \end{pmatrix}$$

where we have used the shorthand

$$R_\mu = [B^2 + (C + D - \lambda_\mu)^2]^{1/2}$$
$$S_{23} = (x^2x^2 + x^3x^3)^{1/2}$$

It is appropriate to change variables. Instead of A, B, C, D, $\{\lambda_\mu\}$, we employ new variables λ, γ, σ, α defined by

$$e^\lambda = -\lambda_0, \qquad e^\gamma = \lambda_1 = \lambda_2, \qquad e^\sigma = \lambda_3$$
$$\sin 2\alpha = -2B[(A - C - D)^2 + 4B^2]^{-1/2}$$
$$\cos 2\alpha = -(A - C - D)[(A - C - D)^2 + 4B^2]^{-1/2}$$

Doubling the angle in the arguments of sine and cosine above avoids having a factor of 1/2 in the expressions for S and Q. The matrices S and Q are given below. The ratio x^i/r is denoted by χ_i; the difference $e^\lambda - e^\sigma$ is denoted by e^-, and $e^\lambda s^2 + e^\sigma c^2 - e^\gamma$ is denoted by F. The functions sin α and cos α are abbreviated, respectively, to s and c. We have

$$S = \begin{pmatrix} e^\lambda c^2 + e^\sigma s^2 & e^- sc\chi_1 & e^- sc\chi_2 & e^- sc\chi_3 \\ e^- sc\chi_1 & e^\gamma + F\chi_1^2 & F\chi_1\chi_2 & F\chi_1\chi_3 \\ e^- sc\chi_2 & F\chi_1\chi_2 & e^\gamma + F\chi_2^2 & F\chi_2\chi_3 \\ e^- sc\chi_3 & F\chi_1\chi_3 & F\chi_2\chi_3 & e^\gamma + F\chi_3^2 \end{pmatrix}$$

$$Q = \begin{pmatrix} -1 + 2s^2 & -2\chi_1 sc & -2\chi_2 sc & -2\chi_3 sc \\ -2\chi_1 sc & 1 - 2\chi_1^2 s^2 & -2\chi_1\chi_2 s^2 & -2\chi_1\chi_3 s^2 \\ -2\chi_2 sc & -2\chi_1\chi_2 s^2 & 1 - 2\chi_2^2 s^2 & -2\chi_2\chi_3 s^2 \\ -2\chi_3 sc & -2\chi_1\chi_3 s^2 & -2\chi_2\chi_3 s^2 & 1 - 2\chi_3^2 s^2 \end{pmatrix}$$

The matrix Q is a function of α but not of λ, γ, σ. The way in which Q represents a member of $SO(3)$ is best seen by introducing the functions $\{\phi_\mu\}$ that define the "hedgehog" of skyrmionic gauge theory (Skyrme, 1961, 1971; Williams, 1970):

$$\phi_0 = \cos\alpha$$

$$\phi_i = \chi_i \sin\alpha = (x^i/r)\sin\alpha, \qquad i = 1, 2, 3$$

Since $\sum \phi_\mu \phi_\mu = 1$, the $\{\phi_\mu\}$ represent (at any instant of time t) a mapping from $\mathbf{R}^3$ into the 3-sphere, S^3. The matrix elements of Q can be written in terms of the $\{\phi_\mu\}$ and the Kronecker delta $\delta_{\mu\nu}$ (Williams, 1971; Williams and Zia, 1973):

$$Q_{\mu\nu} = \delta_{\mu\nu} - 2\phi_\mu\phi_\nu$$

Since the functions ϕ_μ occur in pairs, the mapping determined by Q is a mapping into S^3 *with antipodal points identified*, i.e., a mapping into real projective 3-space $\mathbf{R}P^3$. The latter is homeomorphic to $SO(3)$.

The signature of Q can be displayed by writing Q in the form $Q = V^T \operatorname{diag}(-1, 1, 1, 1)\, V$, where the orthogonal matrix V is given by

$$V = \begin{pmatrix} \phi_0 & \phi_1 & \phi_2 & \phi_3 \\ -\phi_1 & \phi_0 & \phi_3 & -\phi_2 \\ -\phi_2 & -\phi_3 & \phi_0 & \phi_1 \\ -\phi_3 & \phi_2 & -\phi_1 & \phi_0 \end{pmatrix}$$

The metric $g_{\mu\nu}$ can also be expressed in terms of P according to $\|g_{\mu\nu}\| = P \operatorname{diag}(-e^\lambda, e^\gamma, e^\gamma, e^\sigma) P^T$.

Kinks are counted, both in general relativity and in Skyrme's theory, by observing the boundary values of the angle α. For example, if $\alpha(r=0)=\pi$ and $\alpha(\infty)=0$, then the metric has one kink, and this would imply $Q_{\mu\nu}\to\eta_{\mu\nu}$ as either $r\to 0$ or $r\to\infty$. In general, the number of kinks is $N=\{\alpha(0)-\alpha(\infty)\}/\pi$.

If $\alpha\equiv 0$, then $Q=\text{diag}(-1,1,1,1)=\|\eta_{\mu\nu}\|$. Under such circumstances, there would then be no kinks and $g_{\mu\nu}$ would take the usual form that leads to the Schwarzschild solution. A diametrically opposite point of view assumes a trivial $S=\text{diag}(1,1,1,1)=\|\delta_{\mu\nu}\|$ and a nontrivial $Q_{\mu\nu}=\delta_{\mu\nu}-2\phi_\mu\phi_\nu$. This simple kink metric has been studied by Clément (1984*a–c*, 1986), Finkelstein and Williams (1984), Harriott and Williams (1986), 1988*a,b*), Williams (1985), and Williams and Zia (1973). Finkelstein and McCollum (1975) have shown that α is equal to the angle of tilt of the light cones. In the present paper, we allow S and Q to keep their general form, which, putting $\|g_{\mu\nu}\|=SQ$, leads to

$$g_{00}=-e^{\lambda}+(e^{\lambda}+e^{\sigma})\sin^2\alpha$$

$$g_{0i}=g_{i0}=-\chi_i(e^{\lambda}+e^{\sigma})\sin\alpha\cos\alpha$$

$$g_{ij}=\{e^{\sigma}-(e^{\lambda}+e^{\sigma})\sin^2\alpha\}\chi_i\chi_j+e^{\gamma}(\delta_{ij}-\chi_i\chi_j)$$

From now on, it will be convenient to work in spherical polar coordinates t, r, θ, φ, in terms of which the metric $g_{\mu\nu}$ can be written

$$g_{tt}=-e^{\lambda}+(e^{\lambda}+e^{\sigma})\sin^2\alpha$$

$$g_{tr}=g_{rt}=-(e^{\lambda}+e^{\sigma})\sin\alpha\cos\alpha$$

$$g_{rr}=e^{\sigma}-(e^{\lambda}+e^{\sigma})\sin^2\alpha$$

$$g_{\theta\theta}=e^{\gamma}r^2$$

$$g_{\varphi\varphi}=e^{\gamma}r^2\sin^2\theta$$

In the standard approach, a transformation is performed to set $\gamma\equiv 0$. However, for the present we shall keep the above, more general form of metric. Rosen (1983, 1985) has found that, in certain situations, having $\gamma\not\equiv 0$ can lead to new solutions.

3. KINEMATICS AND CURVATURE

In previous work (Harriott and Williams, 1986, 1988*a,b*) where the metric was $Q_{\mu\nu}$ itself, the components u^μ of the 4-velocity were identified with the ϕ_μ (or with its negative). This identification can be generalized and, for the $g_{\mu\nu}$ of the present paper, we now define

$$u^t=e^{-\lambda/2}\cos\alpha,\qquad u^r=e^{-\lambda/2}\sin\alpha,\qquad u^\theta=u^\varphi=0$$

We note that for the $\alpha \equiv 0$ situation, where there are no kinks present, u^t will be the only nonzero component. However, if there is a kink present, there is no nonsingular transformation that will *globally* transform u^r to 0. Of course, locally or within a region of spacetime, it is possible to find a nonsingular transformation that will reduce the velocity vector to the form $(u^t, 0, 0, 0)$, but, after making such a transformation, the simple form of the metric $g_{\mu\nu}$ will be lost.

It now follows that

$$u_t = -e^{\lambda/2} \cos \alpha, \qquad u_r = -e^{\lambda/2} \sin \alpha, \qquad u_\theta = u_\varphi = 0$$

so that the co- and contravariant components are related by

$$u_\mu = -e^{\lambda} u^\mu$$

and $u_\mu u^\mu = -1$, as required.

So far in this paper, the analysis has been quite general, in the sense that the metric may be a function of both r and t. Henceforth, for tractability in computation, we shall assume that the functions λ, γ, σ, α (and hence $g_{\mu\nu}$) depend only on r.

4. TIME-INDEPENDENT SOLUTIONS

With the metric $g_{\mu\nu}$ independent of t, computations will become more manageable and, for the sake of brevity, we introduce the symbols E, Δ, and Ω defined by

$$E = e^{\lambda+\sigma}$$
$$\Delta = 2^{-1}\partial_r\gamma + r^{-1}$$
$$\Omega = \partial_r\Delta - \Delta(2E)^{-1}\partial_r E + \Delta^2$$

The contravariant components $g^{\mu\nu}$, the Christoffel symbols, and the various hydrodynamic quantities (projection tensor h^ν_μ, covariant derivatives $u_{\mu;\nu}$, isotropic expansion θ, shear tensor σ^ν_μ and scalar shear σ) are all listed in the Appendix. Since the metric is spherically symmetric, the components of the vorticity tensor ω^ν_μ are obviously all zero. The computation of the Ricci tensor and the scalar curvature R is straightforward, and these results are also listed in the Appendix. It follows that the nonzero components of the Einstein tensor are

$$G^t_t = -\Delta E^{-1}\{\partial_r g_{tt} + g_{tt}\Delta\} - 2g_{tt}E^{-1}\Omega - e^{-\gamma}r^{-2}$$
$$G^t_r = -2g_{tr}E^{-1}\Omega$$
$$G^r_t = 0$$
$$G^r_r = -\Delta E^{-1}\{\partial_r g_{tt} + g_{tt}\Delta\} - e^{-\gamma}r^{-2}$$
$$G^\theta_\theta = G^\varphi_\varphi = -(2E)^{-2}\{2E\partial^2_r g_{tt} - \partial_r g_{tt}\partial_r E\} - \Delta E^{-1}\partial_r g_{tt} - g_{tt}E^{-1}\Omega$$

Consider the Einstein equations, $G^{\nu}_{\mu} = 8\pi T^{\nu}_{\mu}$, with T^{ν}_{μ} chosen to be the stress-energy tensor for a perfect fluid: $T^{\nu}_{\mu} = (\rho + p)u_{\mu}u^{\nu} + p\delta^{\nu}_{\mu}$. Since $G^{r}_{t} = 0$, the equation $G^{r}_{t} = 8\pi T^{r}_{t}$ implies

$$\rho + p = 0$$

Thus, the stress-energy tensor must have the form $T^{\nu}_{\mu} = p\delta^{\nu}_{\mu}$. Assuming that $\alpha \neq 0$, $G^{t}_{r} = 8\pi T^{t}_{r}$ implies $\Omega = 0$, which (for $\Delta \neq 0$) leads to

$$\Delta^{-1}\partial_r\Delta - (2E)^{-1}\partial_r E + \Delta = 0$$

This can be integrated to give

$$k\Delta e^{\gamma/2} r = E^{1/2}$$

where k is an arbitrary constant. This can be rewritten as

$$\partial_r(r\, e^{\gamma/2}) = k^{-1} e^{(\lambda+\sigma)/2}$$

The condition $\Omega = 0$ simplifies the remaining G^{ν}_{μ} so that

$$G^{t}_{t} = G^{r}_{r} = -\Delta^2 E^{-1}\{k^2 + E^{-1/2}\partial_r(g_{tt}\Delta^{-1}E^{1/2})\}$$

$$G^{\theta}_{\theta} = G^{\varphi}_{\varphi} = \tfrac{1}{2}\Delta\{\partial_r(\Delta^{-2}G^{t}_{t}) + \partial_r(E^{-1})(k^2 + g_{tt} + \Delta^{-1}\partial_r g_{tt})\}$$

The Einstein equations now read

$$G^{t}_{t} = 8\pi p, \qquad G^{\theta}_{\theta} = 8\pi p$$

Suppose that $\gamma \equiv 0$. Then $\Delta = r^{-1}$, and so $E = k$. Little generality is lost in assuming $k = 1$, so that $E = 1$, $\lambda = -\sigma$, and $g_{tt} = -e^{\lambda}\cos^2\alpha + e^{-\lambda}\sin^2\alpha$. The equation for the scalar curvature simplifies to $R = r^{-2}\partial^2_r\{r^2(1+g_{tt})\}$, and the expansion becomes $\theta = r^{-2}\partial_r(r^2 e^{-\lambda/2}\sin\alpha)$. It now follows that

$$G^{t}_{t} = -r^{-2}\{1 + \partial_r(r g_{tt})\} = 8\pi p$$

$$G^{\theta}_{\theta} = (2r)^{-1}\partial_r(r^2 G^{t}_{t}) = 8\pi p$$

These equations imply that p, and hence G^{t}_{t}, is a constant. Since $\rho + p = 0$, we assume p to be a negative constant, $p = -C$. Such constant negative pressures are well known from inflationary models (Guth, 1981; Blau and Guth, 1987). For convenience, we shall also introduce a constant M according to

$$M = (-3/4\pi p)^{1/2} = (3/4\pi C)^{1/2}$$

The G^{t}_{t} equation is

$$\partial_r(r g_{tt}) = 8\pi C r^2 - 1$$

which can be integrated and rewritten as

$$g_{tt} = 8\pi C r^2/3 - 1 + c_1/r$$

where c_1 is an integration constant. Note that putting $\lambda \equiv 0$ gives $g_{tt} = -1 + 2\sin^2\alpha$ and leads to the solution obtained in our previous work (Harriott and Williams, 1988):

$$\sin\alpha = \begin{cases} r/M, & 0 \leq r \leq M \\ (M/r)^{1/2}, & M \leq r < \infty \end{cases}$$

This describes an object of mass and radius M with an exterior vacuum solution. There is positive constant curvature $R = 24/M^2$ and positive constant expansion $\theta = 3/M$. The de Sitter solution has similar properties and it is relatively straightforward (Dunn and Williams, 1989) to construct a transformation between the above solution and the de Sitter solution. However, the transformation is *singular* and so the two solutions describe (globally) different physical systems.

Provided that the equation for g_{tt} is satisfied, there is considerable freedom in choosing $\lambda(r)$. For example, choosing

$$e^{-\lambda} = 1 - 2r^2/M^2$$

leads to

$$\sin^2\alpha = \tanh\lambda \qquad \text{for} \quad 0 \leq r < M/2^{1/2}$$

and to a velocity u^r that tends to zero at the $r = M/2^{1/2}$ boundary. For this solution, $R = 24/M^2$, as before, but $\theta = r^{-2}\partial_r\{r^2(e^{-\lambda}\tanh\lambda)^{1/2}\}$.

Now consider the stress-energy tensor for a viscous fluid (Misner *et al.*, 1973; Ellis, 1971):

$$T^{\nu}_{\mu} = \rho u_\mu u^\nu + (p - \zeta\theta)h^{\nu}_{\mu} - 2\eta\sigma^{\nu}_{\mu} + q_\mu u^\nu + u_\mu q^\nu$$

where the coefficient of bulk viscosity ζ and the coefficient of dynamic viscosity η are both positive functions of ρ and p. For the moment, assume that the heat conduction is negligible: $q_\mu = 0$.

Suppose that $\lambda \equiv 0$ and $\sin\alpha = r/M$ (which was the $\lambda \equiv 0$ solution of the previous section). It is easy to show that the scalar shear is zero and hence $\sigma^{\nu}_{\mu} = 0$, for all μ, ν. Using the above expression for T^{ν}_{μ} and the fact that $\theta = 3/M$, the Einstein equations yield the following equation of state:

$$\rho = -p + 3\zeta/M = C$$

with $C = 3/4\pi M^2$, as before. Thus, ρ is positive and p may also be chosen positive provided that $\zeta > Mp/3$. Since ζ is a function of ρ and p, and both of these are constant, it follows that the bulk viscosity ζ is also a constant. The idea of combining the pressure with the bulk viscosity term has recently been discussed by Pacher *et al.* (1987) in the context of inflation.

Alternatively, let us suppose that $\sigma > 0$, but, for simplicity, that $\zeta = 0$ and $q_\mu = 0$. The equation $G_t^r = 0$ leads to the equation of state:

$$\rho + p - (4/3^{1/2})\eta\sigma = 0$$

Since η is positive, ρ will be positive if $4\eta\sigma/3^{1/2} > p$. Again, $G_r^t = 0$ implies $\Omega = 0$ and the expressions for $G_t^t = G_r^r$ and $G_\theta^\theta = G_\varphi^\varphi$ are the same as those for the perfect fluid. The remaining Einstein equations now read

$$G_t^t = G_r^r = 8\pi\{p - (4/3^{1/2})\eta\sigma\}$$

$$G_\theta^\theta = G_\varphi^\varphi = 8\pi\{p + (2/3^{1/2})\eta\sigma\}$$

As in the perfect fluid case, one might assume $\gamma \equiv 0$, and $E = 1$, so that $\lambda = -\sigma$, $g_{tt} = -e^\lambda \cos^2 \alpha + e^{-\lambda} \sin^2 \alpha$ and the Einstein equations become

$$G_t^t = -r^{-2}\{1 + \partial_r(rg_{tt})\} = 8\pi\{p - (4/3^{1/2})\eta\sigma\}$$

$$G_\theta^\theta = (2r)^{-1}\partial_r(r^2 G_t^t) = 8\pi\{p + (2/3^{1/2})\eta\sigma\}$$

However, it is not clear that one can find a solution with positive energy density. A solution was previously obtained by Harriott and Williams (1988*a*) for the special case of $\lambda = 0$, but this did not have a ρ that was everywhere positive.

Now consider the case where the heat conduction is nonzero: $q_\mu \neq 0$. The condition $q_\mu u^\mu = 0$ suggests

$$q_t = Qe^{\sigma/2} \sin \alpha, \qquad q_r = -Qe^{\sigma/2} \cos \alpha, \qquad q_\theta = q_\varphi = 0$$

where the $e^{\sigma/2}$ factor is included for convenience. Hence

$$q^t = Qe^{-\sigma/2} \sin \alpha, \qquad q^r = -Qe^{-\sigma/2} \cos \alpha, \qquad q^\theta = q^\varphi = 0$$

and Q is seen to be the magnitude of q_μ:

$$q_\mu q^\mu = Q^2$$

The $(q_\mu u^\nu + u_\mu q^\nu)$ term occurring in the expression for T_μ^ν will contain a $e^{\pm(\sigma-\lambda)/2}$ factor. To simplify this, we shall put $\lambda = \sigma$ for the remainder of this section. It follows that

$$q_t u^r + u_t q^r = Q$$

$$q_r u^t + u_r q^t = -Q$$

The Einstein equation $G_t^r = 8\pi T_t^r$ gives

$$Q = \rho + p - \zeta\theta - (4/3^{1/2})\eta\sigma \sin \alpha \cos \alpha$$

The equation $G_r^t = 8\pi T_r^t$ implies that

$$Q = -(8\pi)^{-1}\Omega\, e^{-\lambda} \sin 2\alpha$$

so that nonzero heat conduction requires a nonzero Ω (as well as the presence of the kink). The same Einstein equation leads to the following equation of state:

$$4\pi\{\rho+p-\zeta\theta-(4/3^{1/2})\eta\sigma\}+e^{-\lambda}\Omega=0$$

Since we are now assuming that $\lambda=\sigma$, the expression for E becomes $E=e^{2\lambda}$, so that

$$\Omega=\partial_r\Delta-\Delta\partial_r\lambda+\Delta^2$$

with $\Delta=2^{-1}\partial_r\gamma+r^{-1}$.

Although $\Omega\neq 0$ implies $G^t_t\neq G^r_r$, the two equations $G^t_t=8\pi T^t_t$ and $G^r_r=8\pi T^r_r$ can be shown to be equivalent, after some rearrangement of terms. They can be conveniently expressed as

$$\tfrac{1}{2}(G^t_t+G^r_r)=e^{-\lambda}\Omega+8\pi\{p-\zeta\theta-(4/3^{1/2})\eta\sigma\}$$

The $G^\theta_\theta=8\pi T^\theta_\theta$ equation reads

$$G^\theta_\theta=8\pi\{p-\zeta\theta+(2/3^{1/2})\eta\sigma\}$$

Some assumptions are necessary to simplify the expressions on the left-hand sides of these equations. Set $\lambda=0$, which leads to $E=1$, $g_{tt}=-\cos 2\alpha$. Since we are interested in the situation where $\Omega\neq 0$, we require $\Delta\neq r^{-1}$. Choose $\gamma=2(-kr-\ln r)$, where k is a constant. This gives $\Delta=-k$, $\Omega=k^2$, and $e^{-\gamma}r^{-2}=e^{2kr}$. Let us also assume that the dynamic viscosity η is zero, so that the two Einstein equations become

$$-k\{\partial_r(\cos 2\alpha)-2k\cos 2\alpha\}-e^{2kr}-k^2=8\pi(p-\zeta\theta)$$

and

$$\tfrac{1}{2}\partial_r^2(\cos 2\alpha)-k\partial_r(\cos 2\alpha)+k^2\cos 2\alpha=8\pi(p-\zeta\theta)$$

This suggests the substitution

$$\cos 2\alpha=Ae^{2kr}+B+f(r)$$

where A and B are constants and the, as yet unspecified, function $f(r)$ is included for greater flexibility in seeking a solution with positive energy density. Consistency between the two Einstein equations can be achieved by choosing

$$A=-k^{-2},\qquad B=1,\qquad f(r)=Ce^{\pm 2^{1/2}kr}$$

where C is a constant. For a positive energy density ρ, it turns out that the upper, positive, sign should be chosen in the expression for $f(r)$. From now on, we shall make this choice.

For the interior region, as r increases from 0, we require $\sin\alpha$ to increase from 0 to a maximum value of 1. This is ensured by chosing $C = k^{-2}$:

$$\sin^2\alpha = \tfrac{1}{2}k^{-2}(e^{2kr} - e^{2^{1/2}kr})$$

with $0 \le r \le r_0$, where

$$e^{2kr_0} - e^{2^{1/2}kr_0} = 2k^2$$

Combining the Einstein equations with the equation of state leads to

$$\begin{aligned}\rho &= (8\pi)^{-1}\{-2k\partial_r(\sin^2\alpha) + 4k^2\sin^2\alpha + e^{2kr} - 3k^2\}\\ &= (8\pi)^{-1}\{(2^{1/2}-2)e^{2^{1/2}kr} + e^{2kr} - 3k^2\}\end{aligned}$$

which will be positive for a sufficiently small k, i.e., for $k^2 < (2^{1/2}-1)/3$. Using the formula given in the Appendix, the scalar curvature is found to be

$$R = 2[2\{(2^{1/2}-2)e^{2^{1/2}kr} + e^{2kr}\} - 3k^2]$$

and is positive for the range of values of k that is chosen above.

This interior solution can be connected to an exterior ($r \ge r_0$) solution such as the previously mentioned vacuum solution or a decaying exponential solution, which can be constructed in a similar way to the above increasing exponential solution.

5. SUMMARY

The most general form of spherically symmetric metric has been rewritten in a manner that clearly displays the presence of kinks in terms of the hedgehog function of gauge theory, $(\phi_0, \phi_1, \phi_2, \phi_3) \in S^3$. For a time-independent metric, the curvature, Einstein tensor, and various hydrodynamic quantities were computed and a number of kink solutions were explored. One of these was for an imperfect fluid with nonzero heat conduction.

APPENDIX

The abbreviations

$$E = -g_{rr}g_{tt} + (g_{tr})^2 = e^{\lambda+\sigma}, \qquad \Delta = 2^{-1}\partial_r\gamma + r^{-1}$$

will be used throughout. The determinant and the contravariant components of the metric are

$$\det(g_{\mu\nu}) = -r^4 \sin^2\theta\, e^{\lambda+2\gamma+\sigma}$$

$$g^{tt} = -e^{-\lambda} + (e^{-\lambda} + e^{-\sigma}) \sin^2\alpha$$

$$g^{tr} = g^{rt} = -(e^{-\lambda} + e^{-\sigma}) \sin\alpha \cos\alpha$$

$$g^{rr} = e^{-\sigma} - (e^{-\lambda} + e^{-\sigma}) \sin^2\alpha$$

$$g^{\theta\theta} = (e^{\gamma} r^2)^{-1}$$

$$g^{\varphi\varphi} = (e^{\gamma} r^2 \sin^2\theta)^{-1}$$

The nonzero Christoffel symbols are as follows:

$$\Gamma^t_{tt} = -\Gamma^r_{tr} = -(2E)^{-1} g_{tr} \partial_r g_{tt}$$

$$\Gamma^t_{tr} = -(2E)^{-1} g_{rr} \partial_r g_{tt}$$

$$\Gamma^r_{tt} = (2E)^{-1} g_{tt} \partial_r g_{tt}$$

$$\Gamma^t_{rr} = (2E)^{-1} \{g_{tr} \partial_r g_{rr} - 2 g_{rr} \partial_r g_{tr}\}$$

$$\Gamma^r_{rr} = (2E)^{-1} \{-g_{tt} \partial_r g_{rr} + 2 g_{tr} \partial_r g_{tr}\}$$

$$\Gamma^t_{\theta\theta} = -E^{-1} g_{tr} e^{\gamma} r^2 \Delta$$

$$\Gamma^t_{\varphi\varphi} = \sin^2\theta\, \Gamma^t_{\theta\theta}$$

$$\Gamma^r_{\theta\theta} = E^{-1} g_{tt} e^{\gamma} r^2 \Delta$$

$$\Gamma^r_{\varphi\varphi} = \sin^2\theta\, \Gamma^r_{\theta\theta}$$

$$\Gamma^\theta_{r\theta} = \Gamma^\varphi_{r\varphi} = \Delta$$

$$\Gamma^\theta_{\varphi\varphi} = -\sin\theta \cos\theta$$

$$\Gamma^\varphi_{\theta\varphi} = \cot\theta$$

The components of the projection tensor are

$$h^t_t = \sin^2\alpha$$

$$h^t_r = h^r_t = -\sin\alpha \cos\alpha$$

$$h^r_r = \cos^2\alpha$$

$$h^\theta_\theta = h^\varphi_\varphi = 1$$

The components of the covariant derivative of the velocity vector and the isotropic (volume) expansion θ are

$$u_{t;t}=\tfrac{1}{2}e^{-\lambda/2}\sin\alpha\;\partial_r g_{tt}$$

$$u_{r;t}=-\tfrac{1}{2}e^{-\lambda/2}\cos\alpha\;\partial_r g_{tt}$$

$$u_{t;r}=-\partial_r(e^{\lambda/2}\cos\alpha)+u_{r;t}$$

$$u_{r;r}=-\partial_r(e^{\lambda/2}\sin\alpha)-\tfrac{1}{2}e^{-\lambda/2}(\sin\alpha\;\partial_r g_{rr}+2\cos\alpha\;\partial_r g_{tr})$$

$$u_{\theta;\theta}=e^{-\lambda/2+\gamma}r^2\Delta\sin\alpha$$

$$u_{\varphi;\varphi}=\sin^2\theta\,u_{\theta;\theta}$$

$$u^t_{;t}=-(2E)^{-1}e^{-\lambda/2}\partial_r g_{tt}(g_{tr}\cos\alpha+g_{rr}\sin\alpha)$$

$$u^t_{;r}=\partial_r(e^{-\lambda/2}\cos\alpha)+(2E)^{-1}e^{-\lambda/2}\{-g_{rr}\partial_r g_{tt}\cos\alpha$$
$$+(g_{tr}\partial_r g_{rr}-2g_{rr}\partial_r g_{tr})\sin\alpha\}$$

$$u^r_{;t}=(2E)^{-1}e^{-\lambda/2}\partial_r g_{tt}(g_{tt}\cos\alpha+g_{tr}\sin\alpha)$$

$$u^r_{;r}=\partial_r(e^{-\lambda/2}\sin\alpha)+(2E)^{-1}e^{-\lambda/2}\{g_{tr}\partial_r g_{tt}\cos\alpha$$
$$+(-g_{tt}\partial_r g_{rr}+2g_{tr}\partial_r g_{tr})\sin\alpha\}$$

$$u^\theta_{;\theta}=u^\varphi_{;\varphi}=\Delta e^{-\lambda/2}\sin\alpha$$

$$\theta=u^\mu_{;\mu}=r^{-2}e^{-(\lambda+2\gamma+\sigma)/2}\partial_r(r^2e^{\gamma+\sigma/2}\sin\alpha)$$
$$=e^{(\lambda+\sigma)/2}\partial_r(e^{\sigma/2}\sin\alpha)+2\Delta(e^{-\lambda/2}\sin\alpha)$$

The only nonzero components of the acceleration covector $\dot{u}_\mu=u_{\mu;\nu}u^\nu$ are

$$\dot{u}_t=-e^{-\lambda/2}\sin\alpha\;\partial_r(e^{\lambda/2}\cos\alpha)$$

$$\dot{u}_r=-e^{-\lambda/2}\{\sin\alpha\;\partial_r(e^{\lambda/2}\sin\alpha)-\partial_r(e^{\lambda/2})\}$$

The components of the shear tensor σ^ν_μ and the scalar shear $\sigma=(\sigma^{\mu\nu}\sigma_{\mu\nu}/2)^{1/2}$ are as follows:

$$\sigma=3^{-1/2}\,re^{(-\lambda+\gamma-\sigma)/2}\partial_r\{r^{-1}e^{(-\gamma+\sigma)/2}\sin\alpha\}$$

$$\sigma^t_t=(2/3^{1/2})\sigma\sin^2\alpha$$

$$\sigma^t_r=\sigma^r_t=-(2/3^{1/2})\sigma\sin\alpha\cos\alpha=(2/3^{1/2})\sigma u_t u^r$$

$$\sigma^r_r=(2/3^{1/2})\sigma\cos^2\alpha$$

$$\sigma^\theta_\theta=\sigma^\varphi_\varphi=-3^{-1/2}\sigma$$

The nonzero components of the Ricci tensor are given by

$$R_{tt} = g_{tt}(2E^{1/2})^{-1}\partial_r(E^{-1/2}\partial_r g_{tt}) + 2\Delta\Gamma^r_{tt}$$

$$R_{tr} = g_{tr}(2E^{1/2})^{-1}\partial_r(E^{-1/2}\partial_r g_{tt}) + 2\Delta\Gamma^r_{tr}$$

$$R_{rr} = g_{rr}(2E^{1/2})^{-1}\partial_r(E^{-1/2}\partial_r g_{tt}) + 2\{-\partial_r\Delta + \Delta\Gamma^r_{rr} - \Delta^2\}$$

$$R_{\theta\theta} = 1 + e^{\gamma}r^2E^{-1}[\Delta\partial_r g_{tt} + g_{tt}\{2\Delta^2 + \partial_r\Delta - \Delta(2E)^{-1}\partial_r E\}]$$

$$R_{\varphi\varphi} = \sin^2\theta R_{\theta\theta}$$

The scalar curvature is

$$R = 2E^{-1}\{\Delta(2\partial_r g_{tt} + g_{tt}\Delta) + 2g_{tt}\Omega + Ee^{-\gamma}r^{-2} + \tfrac{1}{2}E^{1/2}\partial_r(E^{-1/2}\partial_r g_{tt})\}$$

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